Surface Integrals, Stokes Theorem

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Derivation/proof of Divergence Thm

Recall for Green's Thm

Say we have a simple closed curve bounding a region A. $C = \partial A$

Thm

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl}\, \vec{F}\, dA$$

Cancellation along overlapping edges

$\Rightarrow$ If $R = R_1 \cup R_2 \cup \cdots \cup R_N$ and let $C_i = \partial R_i$ then

$$\int_C \vec{F} \cdot d\vec{r} = \sum_{i=1}^{N} \int_{C_i} \vec{F} \cdot d\vec{r}$$

$\Rightarrow$ Green's Theorem

Cancellation along overlapping faces for surface integrals

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{\sigma} = \sum_{i=1}^{N} \iint_{S_i} \vec{F} \cdot d\vec{\sigma}$$

$S = \partial R$ is a closed surface

R open connected region in $\mathbb{R}^3$

$R = R_1 \cup R_2 \cup \cdots \cup R_N$

and $S_i = \partial R_i$

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and $S_i = \partial R_i$

this fact is what tells you that

$\iint_S \vec{F} \cdot d\vec{a}$ can be

expressed as a triple integral.

Last time: let R_i be a little cube. Then $S_i = \partial R_i$ has 6 faces in 3 pairs (corresponding to x, y, z)

↳ we showed that the sum of the integrals over the pair of faces in the x direction, ie, $\perp$ to x-axis is $\approx \frac{\partial P}{\partial x} \Delta x \Delta y \Delta z$

$$= \frac{\partial P}{\partial x} \operatorname{vol}(R_i)$$

where $\vec{F} = P\hat{i} + Q\hat{j} + R\hat{k}$

↳ Similarly, can show that the sum of the integrals over the faces in the y-direction is

$$\approx \frac{\partial Q}{\partial y} \operatorname{vol}(R_i)$$

↳ in z-dir:

$$\frac{\partial R}{\partial z} \operatorname{vol}(R_i)$$

$$\Rightarrow \int_{S_i} \vec{F} \cdot d\vec{\sigma} \approx \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) \mathrm{vol}(R_i)$$

gets better as mesh$(\{R_i\}) \to 0$

$$\Rightarrow \iint_S \vec{F} \cdot d\vec{\sigma} = \sum_{i=1}^{N} \iint_{S_i} \vec{F} \cdot d\vec{\sigma}$$

$$\approx \sum_{i=1}^{N} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) \mathrm{vol}(R_i)$$

Now take limit as mesh $\to 0$
(to simplify, think of it as $N \to \infty$)
and get

$$\iint_{\partial R} \vec{F} \cdot d\vec{\sigma} = \iiint_R \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dV$$

divergence thm

For a vector field $\vec{F}$ in $\mathbb{R}^3$:

$$\operatorname{div} \vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Recall for a fcn f on an inverval $[a, b]$, $\dfrac{1}{\mathrm{length}([a,b])} \displaystyle\int_a^b f(x)\,dx$

is the avg value of f on $[a, b]$

Similarly for f defined on a region R in $\mathbb{R}^2$,

$\dfrac{1}{\mathrm{area}(R)} \displaystyle\iint f\, dA =$ average of f on the region R

$$\frac{1}{\text{area}(R)} \iint_R F \, dA = \text{average of } F \text{ on the region } R$$

If R is a region on $\mathbb{R}^3$ and F defined on R,

$$\frac{1}{\text{vol}(R)} \iiint F \, dV = \text{avg of } F \text{ on } R$$

$$\frac{\iiint_{\partial R} \vec{F} \cdot d\vec{\sigma}}{\text{vol}(R)} = \text{avg of div } \vec{F} \text{ on } R$$

Suppose div $\vec{F}$ is const

eg $\vec{F}$ is linear

$$\vec{F} = xy\hat{i} - \frac{y^2}{2}\hat{j} + (xy + 3z)\hat{k}$$

$$\text{div } \vec{F} = 3$$

then

$$\iiint_{\partial R} \vec{F} \cdot d\vec{\sigma} = \text{vo}(R) \cdot C$$

$$\Rightarrow \text{vol}(R) = \frac{\iiint_{\partial R} \left(xy\hat{i} - \frac{y^2}{2}\hat{j} + (xy + 3z)\hat{k}\right) \cdot d\vec{\sigma}}{3}$$

Note: If R really small and div $\vec{F}$ is continuous, then div $\vec{F} \approx$ constant

Try $R = \left\{ \vec{r} \in \mathbb{R}^3 \;\middle|\; \|\vec{r} - \vec{r}_0\| < r \right\}$

= sphere of radius r around $\vec{r}_0$

$$\lim_{r \to 0} \frac{\iint_{\partial R} \vec{F} \cdot d\vec{\sigma}}{\text{vol}(R)} = \text{div}\,\vec{F}(\vec{r}_0)$$

$$= \frac{4}{3}\pi r^3$$

Intuitive / Physical Description of $\iint_S \vec{F} \cdot d\vec{\sigma}$

Recall $\vec{F} \cdot d\vec{\sigma} = \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right) ds\,dt$

$$= \vec{F} \cdot \vec{n}\, dA$$

$$\vec{n} = \frac{\frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t}}{\left\| \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} \right\|}$$

picks out the component of $\vec{F}$ that is perpendicular to S at the given point.

Suppose $\vec{F}$ represents velocity of air at a given point

Suppose there's an open door and let S be the surface bounded by the doorframe

Then $\iint_S \vec{F} \cdot d\vec{\sigma}$ is the rate at which air is transferred btwn the 2 rooms

Say the door is btwn Room A and Room B, then an orientation for S is either

$A \to B$ or
$B \to A$

If we fix orientation to be $A \to B$ then if $\iint_S \vec{F} \cdot d\vec{\sigma}$, it means more air is flowing from $A \to B$ than $B \to A$

If negative, B is losing air, A is gaining air.

Why dot with $\vec{n}$?

If we want to know how much air is flowing from $A \to B$ (or vice versa) → Latin flux → flux integral

We care only about $\vec{F}$ that is $\perp$ to the door

e.g. if the air is flowing in a way $\parallel$ to the door, it shouldn't move between the rooms

Stokes Thm

(basically Green's thm in 3 dim)

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↳ ie, express a line integral in terms of a surface integral $\iint$

Green's thm

$\vec{F} = P\,dx + Q\,dy$

in $\mathbb{R}^2$ viewed as xy-plane in $\mathbb{R}^3$

say $C = \partial R$ for R a 2-D region in $\mathbb{R}^2$

then

$$\int_C \vec{F} \cdot d\vec{r} = \iint_R \text{curl } \vec{F} \cdot dA$$

$$\stackrel{?}{=} \iint_S \vec{g} \cdot \vec{n} \, dA$$

then need: $\text{curl } \vec{F} = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

$= \vec{g} \cdot \vec{n}$

S should just just be R (a subset of $\mathbb{R}^2$)

Bc S is in the xy plane, $\vec{n} = \hat{k}$

↳ (up; NOT $-\hat{k}$ bc

C goes CCW (RHR)

$\Rightarrow$ $\vec{k}$ - component of $\vec{g}$ should be

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Q/ what about other components of $\vec{g}$?

If we want to generalize to 3 dim, want

$$\vec{F} = P\hat{\imath} + Q\hat{\jmath} + R\vec{k}$$

$$\Rightarrow \vec{g} = (\ )\hat{\imath} + (\ )\hat{\jmath} + \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{\text{looks like } \vec{k} \text{ component of cross prod}}\vec{k}$$

$$\nabla \times \vec{F} = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right) \times \left(P\hat{\imath} + Q\hat{\jmath} + R\hat{k}\right)$$

= Vector curl in 3-dim